Explicit Solutions for Safety Problems Using Control Barrier Functions

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Abstract— **The control barrier function approach has been widely used for safe controller synthesis. By solving an online convex quadratic programming problem, an optimal safe controller can be synthesized implicitly. Since the solution is unique, the mapping from the state-space to the control inputs is injective, thus enabling us to evaluate the underlying relationship. In this paper we aim at explicitly synthesizing a safe control law as a function of the state for nonlinear control-affine systems with limited control ability. We transform the online quadratic programming problem into an offline parameterized optimisation problem which considers states as parameters. The obtained explicit safe controller is shown to be a piece-wise Lipschitz continuous function over the partitioned state space if the program is feasible. We address the infeasible cases by solving a parameterized adaptive control barrier function-based quadratic programming problem. Extensive simulation results show the state-space partition and the controller properties.**

I. Introduction

Safety verification and safe controller design for dynamical systems have attracted significant attention in safety critical applications such as collision avoidance, traffic flow control, adaptive cruise control etc. Safety verification aims at verifying that the system trajectory belongs to a safe set over an infinite time horizon. Motivated by invariance analysis [1], safety is proven to be equivalent to set invariance [2]. To establish a relationship between invariant and safe sets, the barrier certificates method has been proposed [3], [4]. This formulation is shown to be efficient for both deterministic and stochastic dynamical system settings; safe controller design remains a challenge, especially for unsafe systems for which the design of a control input is essential for safety. To address this issue, a control barrier function approach was proposed [5].

The control barrier function approach is motivated from the control Lyapunov function approach [6], which guarantees stability of a system by imposing Lyapunov's conditions. Given the system dynamics and a predefined barrier function, safety is guaranteed by redirecting the vector field into the safe set. With an additional relaxation term encoding the distance from the boundary, the safe controller is guaranteed to be Lipschitz continuous [7]. To trade safety and performance, a quadratic programming (QP) formulation is used to integrate control barrier function and control Lyapunov function constraints [5]. This approach was later extended to high-order cases [8], and used in many applications [9].

By considering the states as parameters in the quadratic programming formulation, the optimisation problem turns out to be a multi-parametric one [10]. Major advances have been made in this field for applications related to our problem, such as explicit MPC [11], [12]. To solve this problem, the parameter space is decomposed into critical regions [13]. Inside every critical region, the problem degenerates into an equality-constrained programming problem, whose explicit solution can be evaluated. For the class of quadratic programming and quadratically constrained quadratic programming problems, the critical regions can be enumerated in polynomial time [14].

In this paper we consider the explicit controller design problem for a general nonlinear control-affine dynamical system with limited control ability. We parameterize the state-space, and reformulate the control barrier function-based quadratic programming problem as a multi-parametric programming problem. By analyzing the sensitivity of the optimisation problem at given states (parameters), the state-space is partitioned into multiple critical regions. The safe control law is evaluated explicitly inside every critical region. We further consider the case where the original problem is infeasible for some states. In this case, an alternative adaptive control barrier function formulation is used. Following a similar analysis procedure, we obtain both the safe controller and adapted relaxation term designs as explicit piece-wise functions. Previous work to analyse the explicit closed loop response [15] concentrated on cases without control limitations and did not consider the analysis of Lipschitz continuity and feasibility.

The remainder of the paper is organized as follows. Preliminaries including control barrier functions and sensitivity analysis are introduced in Section II. The explicit controller design approach is introduced in Section III. Numerical simulation results are reported in Section IV. Section V concludes the paper.

II. Preliminaries and Notations

Notation: For matrix A, we use A_i to represent the i-th row of A. For an index set $\mathcal{I}$, $A_{\mathcal{I}}$ denotes the matrix whose rows are $A_i, i \in \mathcal{I}$. We use $A \succeq 0$ and $A \succ 0$ to denote that A is positive semi-definite and positive definite, respectively. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ is the positive half set, and $\mathbb{R}^m_+$ lifts the dimension to m.

A. Control barrier functions

Consider a nonlinear control-affine system

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, and $g(x) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Both functions are further assumed to be locally Lipschitz continuous. Defining the solution of (1) to be $\psi(u, t, x_0)$, where x_0 represents the initial

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condition and t denotes time, our goal is to design a closed loop controller $u(x)$ so that $\psi(u(x), t, x_0)$ stays within a given safe set $\mathcal{S}$ for any t. The existence and uniqueness of the solution $\psi(u(x), t, x_0)$ are guaranteed by assuming the system is *forward complete*, i.e. $\psi(u(x), t, x_0)$ is unique for any $t \geq 0$.

The control barrier function approach answers the question of how to design a closed loop safe controller $u(x)$ inside a region $\mathcal{B} \subseteq \mathbb{R}^n$, and how to guarantee the resulting safe controller is Lipschitz continuous. The notion of control barrier functions is facilitated by the notion of extended class-$\mathcal{K}$ functions.

Definition 1. *A continuous function $\alpha(\cdot) : (-b, a) \to (-\infty, +\infty)$ is said to be an extended class-$\mathcal{K}$ function for positive a and b, if it is strictly increasing and $\alpha(0) = 0$.*

Definition 2. *For the control-affine dynamical system* (1)*, a continuously differentiable function $B(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is said to be a control barrier function for the set $\mathcal{B}$, if there exists an extended class-$\mathcal{K}$ function $\alpha(\cdot)$ and a set $\mathcal{C}$, where $\mathcal{B} \subseteq \mathcal{C} \subset \mathbb{R}^n$, such that for any $x \in \mathcal{C}$,*

$$\sup_{u \in \mathcal{U}} [\mathcal{L}_f B(x) + \mathcal{L}_g B(x) u + \alpha(B(x))] \geq 0. \tag{2}$$

Here $\mathcal{L}_f B(x)$ and $\mathcal{L}_g B(x)$ are Lie derivatives, which are defined by $\mathcal{L}_f B(x) := \frac{\partial B(x)}{\partial x} f(x)$ and $\mathcal{L}_g B(x) := \frac{\partial B(x)}{\partial x} g(x)$, respectively.

Given a control barrier function $B(x)$, the control admissible set corresponding to (2) is defined by

$$K_{cbf}(x) := \{u \in \mathcal{U} : \mathcal{L}_f B(x) + \mathcal{L}_g B(x) u + \alpha(B(x)) \geq 0\}. \tag{3}$$

Lemma 1. *[5] Consider a control barrier function $B(x)$ defined on $\mathcal{C}$. Then for any $x \in \mathcal{C}$, any $u(x) \in K_{cbf}(x)$ renders the set $\mathcal{B}$ forward invariant.*

The set $K_{cbf}(x)$ maps state x to control input $u(x)$. This motivates us to compute the explicit expression of the closed-loop controller, possibly compromising performance.

B. Sensitivity Analysis

Consider the following multi-parametric optimisation problem

$$\begin{aligned} &\min_x \ f(x, \theta) \\ &\text{subject to } g_i(x, \theta) \leq 0, \forall i \in \mathbb{I}, h_j(x, \theta) = 0, \forall j \in \mathbb{J}, \end{aligned} \tag{4}$$

where $\theta \in \mathcal{W} \subseteq \mathbb{R}^v$ is a parameter. Suppose the functions $f(x, \theta), g_i(x, \theta), h_j(x, \theta), \forall i \in \mathbb{I}, \forall j \in \mathbb{J}$ are convex in x. The problem here is to represent the optimizer of (4) as a function of θ, i.e. $x^*(\theta)$. To do this, one strategy is to solve the multi-parametric optimisation problem for different parameters θ^*. Then the problem turns out to be a general optimisation problem. In a small region around θ^*, $x^*(\theta)$ has an identical expression. The readers are referred to [13] for more information on this topic.

III. Explicit Safe Controller Design

We now show how to synthesize an explicit safe controller by evaluating the explicit solution to the control barrier function-based quadratic programming problem. Suppose that for system (1) and a set $\mathcal{C}$, there exists a control barrier function $B(x)$, with $\mathcal{B}$ defined by its zero super-level set.

A. Control barrier functions based QP

To design a safe controller satisfying the control barrier functions constraint (2), a QP is used to search for feasible points and to guarantee controller performance. Usually, a nominal controller $u^{\text{des}}(x) \in \mathcal{U}$ is synthesized with other methods in advance, e.g. optimal control or PID, for controller performance. The goal for the quadratic programming setting is to find a safe controller $u^*(x)$ with a minimized bias from $u^{\text{des}}(x)$. The QP is given in the following form:

$$\begin{aligned} u^*(x) = \arg\min_u \ & \frac{1}{2}||u - u^{\text{des}}(x)||_2^2 \\ \text{subject to } & \mathcal{L}_f B(x) + \mathcal{L}_g B(x) u + \alpha(B(x)) \geq 0, \\ & Au + b \leq 0, \end{aligned} \tag{5}$$

where $A \in \mathbb{R}^{p \times m}$ and $b \in \mathbb{R}^p$. Here $Au + b \leq 0$ represents the control input constraints. The Lagrangian of problem (5) is

$$\begin{aligned} L(u, \lambda, \mu) = & \frac{1}{2}||u - u^{\text{des}}||_2^2 + \mu^\top (Au + b) \\ & - \lambda(\mathcal{L}_f B(x) + \mathcal{L}_g B(x) u + \alpha(B(x))), \end{aligned} \tag{6}$$

where $\lambda \in \mathbb{R}_+, \mu \in \mathbb{R}_+^p$ are the Lagrange multipliers.

Clearly, in (2) the objective function is strongly convex, and the constraints are linear, thus strong duality holds. The optimal controller $u^*(x)$ can be derived by applying the Karush-Kuhn-Tucker (KKT) conditions

$$u^*(x) - u^{\text{des}} + G(x)^\top \lambda + A^\top \mu = 0, \tag{7a}$$

$$\mu^\top (Au^*(x) + b) = 0, \tag{7b}$$

$$\lambda(F(x) + G(x) u^*(x) + \Lambda(x)) = 0, \tag{7c}$$

$$\lambda \geq 0, F(x) + G(x) u^*(x) + \Lambda(x) \leq 0, \tag{7d}$$

$$\mu \geq 0, Au^*(x) + b \leq 0. \tag{7e}$$

Here we substitute $-\mathcal{L}_f B(x)$, $-\mathcal{L}_g B(x)$, $-\alpha(B(x))$ for $F(x)$, $G(x)$, $\Lambda(x)$, respectively. Though λ, μ are also functions of parameter x, we omit the argument without ambiguity. The reason we are interested in the KKT condition is that, at a given state, the optimal control law $u^*(x)$ fulfills the equality conditions in (7a) – (7c), which enables us to explore explicit expressions. On the other hand, the inequality conditions (7d) – (7e) specify optimality and feasibility conditions, which define critical regions.

To ease exposition, we assume that the relative degree of the control barrier function $B(x)$ is one. The results of this paper can be extended to high order cases. Moreover, the primal degenerate cases where linearly dependent constraints exist, and dual degenerate cases where $\lambda = 0$ or $\mu = 0$, are omitted. These cases are addressed in Section III-B via a pruning scheme.

For a given x, we separate the control admissible constraints into active constraints $A_i u^*(x)+b_i=0$, and inactive ones $A_j u^*(x)+b_j<0$. The active index set $\mathcal{I}(x)$ is defined by $\mathcal{I}(x)=\{i|A_i u^*(x)+b_i=0\}$, while the index set of inactive constraints is denoted by $\bar{\mathcal{I}}(x)$. At a given x, the index set can be verified by testing all the control admissible constraints. With such a separation scheme, we are able to evaluate the solution of (4) from three cases by considering whether the control barrier function constraint is active or not.

Case 1: Given x, the control barrier function constraint is inactive, i.e. $F(x)+G(x)u+\Lambda(x)<0$. The optimizer of (4) in this case is given by

$$u^*(x)=u^{\mathrm{des}}. \tag{8}$$

To see this, we have already assumed that $u^{\mathrm{des}}\in\mathcal{U}$. The saddle point of the primal function $\frac{1}{2}||u-u^{\mathrm{des}}||_2^2$ is u^{des}, which is a feasible point satisfying the constraint $Au+b\le 0$. Hence, the minimizer of problem (4) when the control barrier function constraint is inactive, is u^{des}.

Case 2: Given x, the control barrier function constraint is active, i.e. $F(x)+G(x)u(x)+\Lambda(x)=0$. The control admissible constraints are all inactive i.e. $\mathcal{I}(x)=\emptyset$ and $\mu=0$. Suppose that the optimisation problem (4) is feasible, then the optimizer $u^*(x)$ and multiplier $\lambda(x)$ in this case are

$$u^*(x)=u^{\mathrm{des}}-\frac{G(x)^\top(F(x)+G(x)u^{\mathrm{des}}+\Lambda(x))}{G(x)G(x)^\top}, \tag{9a}$$

$$\lambda(x)=\frac{F(x)+G(x)u^{\mathrm{des}}+\Lambda(x)}{G(x)G(x)^\top}. \tag{9b}$$

To see this, by substituting $\mu=0$ into condition (7a) we have:

$$u^*(x)=u^{\mathrm{des}}-G(x)^\top\lambda(x). \tag{10}$$

Here $\lambda(x)\neq 0$ since the control barrier function constraint is active. Substitute $u^*(x)$ from (10) into $F(x)+G(x)u^*(x)+\Lambda(x)=0$ to get (9b), where $G(x)G(x)^\top\neq 0$ provided that the relative degree of the control barrier function is one. We can then substitute $\lambda(x)$ from (9b) into (10) to obtain (9a).

Case 3: Given x, the control barrier function constraint is active, i.e. $F(x)+G(x)u+\Lambda(x)=0, \lambda(x)>0$, the control limit constraints are active with the index set $\mathcal{I}(x)$. Suppose that the problem (4) is feasible, then the optimizer and multipliers of (6) in this case are

$$u^*(x)=u^{\mathrm{des}}-G(x)^\top\lambda(x)-A_{\mathcal{I}}{}^\top\mu(x), \tag{11a}$$

$$\begin{aligned}\lambda(x)=&(G(x)G(x)^\top)^{-1}\\ &\times(F(x)+G(x)u^{\mathrm{des}}-G(x)A_{\mathcal{I}}\mu+\Lambda(x)),\end{aligned} \tag{11b}$$

$$\begin{aligned}\mu(x)=&(A_{\mathcal{I}}\tilde{G}^+(x)A_{\mathcal{I}}{}^\top)^{-1}\\ &\times(A_{\mathcal{I}}\tilde{G}^-(x)u^{\mathrm{des}}-A_{\mathcal{I}}G(x)^\top(G(x)G(x)^\top)^{-1}F(x)-\\ &A_{\mathcal{I}}G(x)^\top(G(x)G(x)^\top)^{-1}\Lambda(x)+b_{\mathcal{I}}),\end{aligned} \tag{11c}$$

where $\tilde{G}^-(x):=I-G(x)^\top(G(x)G(x)^\top)^{-1}G(x)$, $\tilde{G}^+(x):=I+G(x)^\top(G(x)G(x)^\top)^{-1}G(x)$.

The results are computed by applying the KKT condition (7a) and the active constraint equations at the saddle point $(u^*(x),\lambda(x),\mu(x))$; we have $u^*(x)-u^{\mathrm{des}}+G(x)^\top\lambda(x)+A_{\mathcal{I}}{}^\top\mu(x)=0$, $F(x)+G(x)u^*(x)+\Lambda(x)=0$, $A_{\mathcal{I}}u^*(x)+b_{\mathcal{I}}=0$. Solving the above linear equations, we get the expressions in (11). Given that $G(x)$ is full row rank, thus $\tilde{G}^+(x)$ is also full row rank, which leads to the existence of $(A_{\mathcal{I}}\tilde{G}^+(x)A_{\mathcal{I}}{}^\top)^{-1}$ provided that $A_{\mathcal{I}}$ is full row rank.

Clearly, the aforementioned three cases cover all the possible situations when solving problem (4). The following theorem states the Lipschitz continuity property of the optimal safe control input $u^*(x)$.

Theorem 1. *Consider problem* (4)*. Suppose that this problem is feasible for any $x\in\mathcal{C}$. Then the optimal control law $u^*(x)$ is locally Lipschitz continuous in $\mathcal{C}$.*

Proof. The multipliers $\lambda(x),\mu(x)$, and control law $u^*(x)$ are bounded because of the relative degree assumption of $B(x)$. Hence, $u^*(x)$ is Lipschitz continuous within every domain in the three cases with expressions (8), (9), (11), since the composition and product of locally Lipschitz continuous functions are Lipshitz continuous. Besides, $u^*(x)$ is well defined and continuous on the boundary of the domain provided that all the constraints satisfy strict complementary slackness and are linearly independent [11]. Hence, we conclude that the optimal safe controller $u^*(x)$ is locally Lipschitz continuous over $\mathcal{C}$. □

Equations (8) – (11) characterize the solutions only in a neighborhood of a specific x. Given $x\in\mathcal{C}$, the question remaining is to which case this belongs to. The following section provides an approach to explore the state-space.

B. Partitioning the state space

After presenting the explicit solution evaluation for all possible cases in the previous section, we now show how to partition the state space and how to address the primal degenerate and dual degenerate cases.

The exploration of the parameter space has been widely investigated in the literature. Our method lies in the class of active-set approaches [16], which tend to enumerate all possible active-inactive combinations of the constraints. The state-space is partitioned into a finite number of critical regions.

The proposed approach is based on the enumeration of all possible combinations of active and inactive control limit constraints. Suppose there are overall p constraints. Let $\mathcal{AS}$ denote the set of active sets, and $\overline{\mathcal{AS}}$ denote the inactive sets

$$\mathcal{AS}:=\{\mathcal{I}_1=\{1\},\ldots,\mathcal{I}_{2^p-1}=\{1,\ldots,p\}\}, \tag{12a}$$

$$\overline{\mathcal{AS}}:=\{\bar{\mathcal{I}}_1=\{2,\ldots,p\},\ldots,\bar{\mathcal{I}}_{2^p-1}=\emptyset\}. \tag{12b}$$

Our approach is shown in Algorithm 1. Critical regions for Case 1 and Case 2 are partitioned in lines 1-2, denoted by CR_1 and CR_2, respectively. If the CBF constraint is active, then we enumerate all possible index sets, and evaluate the controller $u(x)$ by Case 3. Lines 5-6 deal with the linearly dependent constraints. If $A_{\mathcal{I}}$ is not full row rank, this directly indicates that there are linear dependent constraints. This case can be pruned since there are redundant linear equations involved. Line 8 corresponds to Case 3. We note here that in the algorithm, strict inequalities are used to partition the state space, and therefore the boundary of each region is not well defined. Since $u^*(x)$ is continuous on every boundary, one can assign the limit points to any neighbouring region. More specifically, the boundary corresponds to those x where the optimisation problem (4) is dual degenerate.

Algorithm 1: State space partition algorithm

Input: Matrix A, b, functions $G(x), F(x), \Lambda(x)$, vector u^{des}

Output: Optimal control law $u^*(x)$, critical regions $\text{CR}_1, \ldots, \text{CR}_l$

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If the control barrier function constraint is inactive, obtain
  the critical region by substituting the parametric
  expressions on the inactive control barrier function
  constraint
  CR_1 := C ∩ {x | F(x) + G(x)u*(x) + Λ(x) < 0}, where
  u*(x) is given by (8)
If the control barrier function constraint is active and the
  control limits constraints are inactive, the critical region is
  defined by
  CR_2 := C ∩ {x | Au*(x) + b < 0} ∩ {x | λ(x) > 0}},
  where u*(x) and λ(x) are given by (9).
If the control barrier function constraint is active,
  enumerate all possible pair-wise combinations of active
  sets AS and inactive sets \overline{AS}.
for i=1:p do
  if A_{I_i} is not full row rank then
    Prune this case
  else
    The critical region is defined by
      CR_{i+2} := C ∩ {x | A_{\bar{I}_i} u*(x) + b_{\bar{I}_i} < 0}
      ∩{x | λ(x) > 0} ∩ {x | μ(x) > 0}, where u*(x),
      λ(x), μ(x) are given by (11).
  end
end
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C. Adaptive control barrier function-based QP

In this section, we consider the feasibility problem discussed in the previous section. Lipschitz continuity of the controller is guaranteed by the relaxation term, i.e. the class-$\mathcal{K}$ function $\alpha(\cdot)$ in (2). At some $x \in \mathcal{C}$, the problem could be infeasible due to the control limits or unsuitable $\alpha(\cdot)$. Numerous adaptive control barrier function approaches have been proposed to tune $\alpha(\cdot)$ dynamically to improve feasibility [17]. However, in these results the explicit adapted relaxation term design is not revealed, neither the Lipschitz continuity of $u^*(x)$ is shown. In this section we address these problems by solving an adaptive control barrier function-based QP problem explicitly.

The adaptive control barrier function-based QP can be formulated as

$$\begin{aligned}(s^*(x), u^*(x)) = \arg\min_{s,u} \ & \frac{p_s}{2}(s-1)^2 + \frac{1}{2}||u - u^{\text{des}}||_2^2 \\ \text{subject to } & F(x) + G(x)u + s\Lambda(x) \leq 0, \\ & Au + b \leq 0, \end{aligned} \tag{13}$$

where s is an adapted parameter to tune the relaxation term dynamically, and $p_s \in \mathbb{R}^+$ is a predefined penalty coefficient. The Lagrangian of the problem (13) is

$$\begin{aligned} L(s, u, \lambda, \mu) =& \frac{p_s}{2}(s-1)^2 + \frac{1}{2}||u - u^{\text{des}}||_2^2 + \\ & \mu^\top(Au + b) + \lambda^\top(F(x) + G(x)u + s\Lambda(x)). \end{aligned} \tag{14}$$

For every x the problem is a convex QP over both s and u, strong duality holds. The KKT conditions for the optimisation problem are

$$p_s s^*(x) - p_s + \Lambda(x)\lambda = 0, \tag{15a}$$
$$A^\top \mu + G(x)^\top \lambda + u^*(x) - u^{\text{des}} = 0, \tag{15b}$$
$$\mu^\top(Au^*(x) + b) = 0, \tag{15c}$$
$$\lambda(F(x) + G(x)u^*(x) + s^*(x)\Lambda(x)) = 0, \tag{15d}$$
$$\lambda \geq 0, F(x) + G(x)u^*(x) + s^*(x)\Lambda(x) \leq 0, \tag{15e}$$
$$\mu \geq 0, Au^*(x) + b \leq 0. \tag{15f}$$

Following the steps in Section III-A, the problem can be solved by considering three cases. One difference here is that the feasibility requirement is no longer assumed when solving the problem explicitly.

Theorem 2. *The optimisation problem* (13) *is feasible for any $x \in \mathcal{C}$, and any class-$\mathcal{K}$ function $\Lambda(x)$ if $B(x)$ is a control barrier function.*

Proof. We first prove the theorem for $\Lambda(x) > 0$ and then $\Lambda(x) = 0$. For any $x' \in \{x|\Lambda(x) > 0\} \subseteq \mathcal{C}$, $u' \in \mathcal{U}$ the search space $s \leq -\frac{F(x)+G(x)u'}{\Lambda(x)}$ is nonempty since $-\frac{F(x)+G(x)u'}{\Lambda(x)}$ is a finite real scalar. Thus, problem (13) is feasible. Consider now the case where $x' \in \{x|\Lambda(x) = 0\} \subseteq \mathcal{C}$. According to the definition of control barrier functions, we have $\sup_{u\in\mathcal{U}}[\mathcal{L}_f B(x) + \mathcal{L}_g B(x)u + 1 \cdot \Lambda(x)]|_{x=x'}$, the problem (13) is feasible with $s^*(x) = 1$. Therefore, we conclude that for any $x \in \mathcal{C}$, and any class-$\mathcal{K}$ function $\Lambda(x)$, the adaptive control barrier function based QP problem (13) is feasible. □

Remark. *For the case where $\mathcal{B} \subseteq \mathcal{S}$ but $B(x)$ is not a proper control barrier function,* (13) *is feasible for any $x \in \text{Int}(\mathcal{B})$. For somer $x \to \partial\mathcal{B}$, we directly have that $s^*(x) \to \infty$ if $F(x) + G(x)u < 0$ for any $u \in \mathcal{U}$, since $\Lambda(x) \to 0$.*

Case 1: Given x, the control barrier function constraint is inactive i.e. $F(x) + G(x)u^*(x) + s\Lambda(x) < 0$ and $\lambda = 0$. The minimized adapted parameter $s^*(x)$ of (13) in this case is given by

$$s^*(x) = 1. \tag{16}$$

The result follows directly from (15a) by substituting $\lambda = 0$.

Case 2: Given x, the control barrier function constraint is active, and the control limit constraints are inactive, i.e. $Au + b = 0$ and $\mu = 0$. Then the optimal control law $u^*(x)$, the minimized adapted parameter $s^*(x)$, and the Lagrange multiplier $\lambda(x)$ are given by

$$u^*(x) = u^{\text{des}} - G(x)^\top \lambda(x), \tag{17a}$$
$$s^*(x) = 1 - \frac{\Lambda(x)}{p_s}\lambda(x), \tag{17b}$$
$$\lambda(x) = \frac{p_s(\Lambda(x) + G(x)u^{\text{des}} + F(x))}{p_s G(x)G(x)^\top + \Lambda(x)^2}. \tag{17c}$$

To prove this, let $\mu = 0$ into (15b), we obtain linear equations. Solving these linear equations leads to (17).

From the results for Case 2 we can see that the adapted parameter $s^*(x) = 1$ only when $\Lambda(x) + G(x)u^{\text{des}} + F(x) = 0$, which shows that $\lambda(x) = 0$. Therefore, we conclude that the adapted parameter needs to be tuned if and only if the control barrier function constraint is active.

Case 3: Given x, the control barrier function constraint is active and the control limit constraints are active with index set $\mathcal{I}$. Then the optimal adapted parameter $s^*(x)$, optimal control input $u^*(x)$ and the Lagrange multipliers $\lambda(x), \mu(x)$ are given by

$$s^*(x) = 1 - \frac{\Lambda(x)}{p_s}\lambda(x), \tag{18a}$$

$$u^*(x) = u^{\text{des}} - A_{\mathcal{I}}^\top \mu(x) - G(x)^\top \lambda(x), \tag{18b}$$

$$\mu(x) = (A_{\mathcal{I}} A_{\mathcal{I}}^\top)^{-1}(A_{\mathcal{I}} u^{\text{des}} - A_{\mathcal{I}} G(x)^\top \lambda(x) + b), \tag{18c}$$

$$\begin{aligned} \lambda(x) = {} & (p_s G(x) \tilde{A}_{\mathcal{I}}^- G(x)^\top + \Lambda(x)^2)^{-1} \\ & \times p_s (F(x) + G(x) \tilde{A}_{\mathcal{I}}^- u^{\text{des}} + \Lambda(x) \\ & - G(x) A_{\mathcal{I}}^\top (A_{\mathcal{I}} A_{\mathcal{I}}^\top)^{-1} b_{\mathcal{I}}), \end{aligned} \tag{18d}$$

where $\tilde{A}_{\mathcal{I}}^- := I - A_{\mathcal{I}}^\top (A_{\mathcal{I}} A_{\mathcal{I}}^\top)^{-1} A_{\mathcal{I}}$, with the observation that $p_s G(x) \tilde{A}_{\mathcal{I}}^- G(x)^\top + \Lambda(x)^2$ is non-zero, since $\tilde{A}_{\mathcal{I}}^- \succeq 0$ and $p_s > 0$, $\Lambda(x) \geq 0$.

Having listed all the possible cases for explicit adapted parameter design, the region characteristic for this problem follows similar steps as in Algorithm 1. Ideally, $s^*(x) = 1$ in the feasible critical regions for the original problem (4). However, this cannot be achieved unless we have $p_s \to \infty$.

The following theorem states the Lipschitz continuity proof for controller design with adapted formulation.

Theorem 3. *The optimal control law $u^*(x)$ obtained by solving* (13) *is locally Lipschitz continuous in $\mathcal{C}$.*

Proof. The proof is similar to that of Theorem 1, since $s^*(x)$ is locally Lipschitz continuous in every critical region, and is continuous on the boundary. □

The new relaxation term $s^*(x)\alpha(B(x))$ has no obvious monotonicity property according to (17b) and (18a). In fact any locally Lipschitz continuous relaxation function $\alpha(x)$ values zero for any $x \in \{x | B(x) = 0\}$ guarantees safety as well as renders the controller locally Lipschitz continuous.

IV. Simulation Results

We first illustrate the state space exploration for explicit safe controller synthesis. Consider the following linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{19}$$

with affine control limits $-1 \leq u_1 \leq 1$, $-1 \leq u_2 \leq 1$. A quadratic control barrier function is defined by $B(x) = -x_1^2 - x_2^2 + 9$. The class-$\mathcal{K}$ function is $\alpha(B(x)) = 0.5B(x)$, and $u^{\text{des}} = [0.5, 0.5]^\top$. Following the state-space partitioning procedure in Algorithm 1, we obtain the critical regions listed in Table I. We observe that the active index set $\mathcal{I}$ contains at most one element, in this case due to the linear independence requirement.

As this problem is infeasible close to the boundary of the safe region, we can use the adaptive control barrier function formulation (13) to improve feasibility. With a larger coefficient, the control barrier function constraint is relaxed and the problem is rendered solvable if $B(x)$ is a candidate control barrier function according to Theorem 2. Following the same procedure as in Algorithm 1, the state space partitioning is shown in Table II. Figure 2 shows the domain partition, values of $u^*(x)$ and $s^*(x)$. It can be seen that the problem is feasible for any $x \in \partial\mathcal{B}$ in Figure 2(a). Figures 2(b)-2(c) show that $u^*(x)$ is locally Lipschitz continuous with the adaptive control barrier function formulation. In Figure 2(d), note that $s^*(x) \to \infty$ when $x \to \partial\mathcal{B}$. This indicates that $B(x)$ is not a candidate control barrier function for system (19) with control input limits.

Figure 1(a) shows the partitioned regions. The safe region is partitioned into seven critical regions, in which $u^*(x)$ is defined as a piece-wise continuous function. The problem

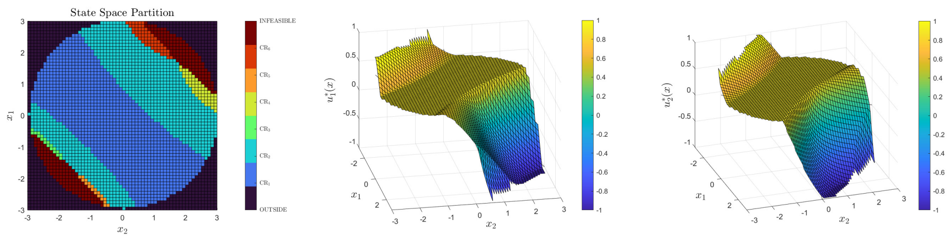
Critical Regions	Conditions Activeness	Index Set $\mathcal{I}$
CR1	$F(x) + G(x)u + \Lambda(x) < 0$*	$\emptyset$
CR2	$F(x) + G(x)u + \Lambda(x) = 0$	$\emptyset$
CR3	$F(x) + G(x)u + \Lambda(x) = 0$ $u_1(x) = 1$	[1]
CR4	$F(x) + G(x)u + \Lambda(x) = 0$ $u_1(x) = -1$	[2]
CR5	$F(x) + G(x)u + \Lambda(x) = 0$ $u_2(x) = 1$	[3]
CR6	$F(x) + G(x)u + \Lambda(x) = 0$ $u_2 = -1$	[4]
CR7	NaN**	NaN

* The control barrier function constraint is inactive (Case 1)
** The problem is infeasible

TABLE I
State space partitioning for the control barrier functions based quadratic programming

Critical Regions	Conditions Activeness	Index Set $\mathcal{I}$
CR1	$F(x) + G(x)u + s\Lambda(x) < 0$	$\emptyset$
CR2	$F(x) + G(x)u + s\Lambda(x) = 0$	$\emptyset$
CR3	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1(x) = 1$	[1]
CR4	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1(x) = -1$	[2]
CR5	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_2(x) = 1$	[3]
CR6	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_2 = -1$	[4]
CR7	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1 = 1$ $u_2 = 1$	[1,3]
CR8	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1 = 1$ $u_2 = -1$	[1,4]
CR9	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1 = -1$ $u_2 = 1$	[2,3]
CR10	$F(x) + G(x)u + s\Lambda(x) = 0$ $u_1 = -1$ $u_2 = -1$	[2,4]

TABLE II
State space partitioning for the adaptive control barrier functions based quadratic programming

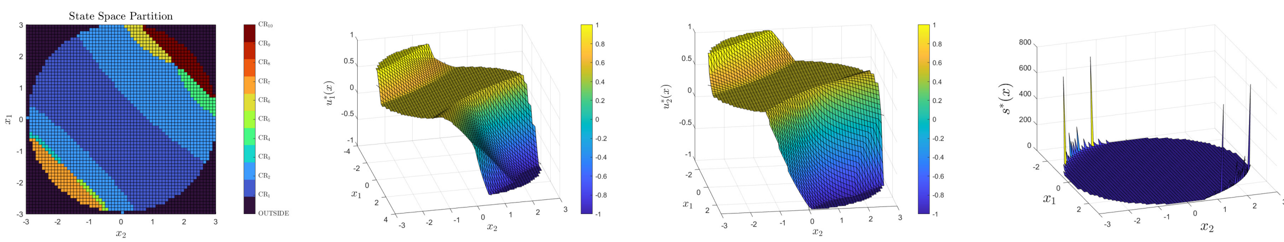


(a) domain partition for $u^*(x)$ derived from (2)

(b) value of $u_1^*(x)$

(c) value of $u_2^*(x)$

Fig. 1. Value of $u^*(x)$ in the state space, the piece-wise explicit expression is Lipschitz continuous



(a) domain partition for $u^*(x)$ derived from (2)

(b) value of $u_1^*(x)$

(c) value of $u_2^*(x)$

(d) value of $s^*(x)$

Fig. 2. Value of $u^*(x)$ in the state space, the piece-wise explicit expression is Lipschitz continuous

is infeasible inside some regions of the safe region which suggests that either $B(x)$ is not a candidate control barrier function, or the $\alpha(B(x)) = 0.5B(x)$ is not a proper relaxation function in class-$\mathcal{K}$. Figures 1(b)-1(c) show the value of $u_1^*(x)$ and $u_2^*(x)$ in the state space. In CR_1, $u^*(x) = u^{\text{des}}$ has a constant value. Overall $u^*(x)$ is locally Lipschitz continuous.

V. Conclusion

In this paper we investigated the explicit safe controller synthesis problem. The proposed approach was based on parameterized control barrier functions-based quadratic programming. Exploring the state space, multiple disjoint critical regions are identified as domains for piece-wise explicit controller design. For the case where the problem is infeasible we propose an adaptive coefficient adaptation scheme. Simulation results demonstrate our results. Future work aims at investigating how to enhance the smoothness of the resulting controller with carefully designed relaxation terms.

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